

Two types of approximate identities depending on the character spaces of Banach algebras

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Abstract

Let A be a commutative Banach algebra with non-empty character space $\Delta(A)$. In this paper, we give two notions of approximate identities for A . Indeed, we change the concepts of convergence and boundedness in the classical notion of bounded approximate identity. More precisely, a net $\{e_\alpha\}$ in A is a *c-w approximate identity* if for each $a \in A$, the Gel'fand transform of $e_\alpha a$ tends to the Gel'fand transform of a in the compact-open topology and we say $\{e_\alpha\}$ is *weakly bounded* if the image of $\{e_\alpha\}$ under the Gel'fand transform is bounded in $C_0(\Delta(A))$.

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1 Introduction

In 1977, Jones and Lahr introduced a new notion of an approximate identity for a commutative Banach algebra A called "bounded weak approximate identity (b.w.a.i)" and gave an example of a semisimple and commutative semigroup algebra (semisimplicity implies that its character space is non-empty) with a b.w.a.i which had no approximate identity, bounded or unbounded. Indeed, a bounded net $\{e_\alpha\}$ in A is a b.w.a.i, if for each $a \in A$ and $\phi \in \Delta(A)$, $|\widehat{ae_\alpha}(\phi) - \widehat{a}(\phi)| \rightarrow 0$. Since $\Delta(A)$ is a locally compact Hausdorff space, one may ask the following question: What happens if the net $\{e_\alpha\}$ satisfies the following condition(s):

For each $a \in A$ and compact subset K of $\Delta(A)$, $\sup_{\phi \in K} |\widehat{ae_\alpha}(\phi) - \widehat{a}(\phi)| \rightarrow 0$, and what happens if we borrow the boundedness of $\{e_\alpha\}$ from $C_0(\Delta(A))$ in the sense that there exists an $M > 0$ such that for each α , $\|\widehat{e_\alpha}\|_{\Delta(A)} < M$?

In this paper, we investigate the above question(s) for commutative Banach algebras with non-empty character spaces.

2 Preliminaries

Throughout the paper, let A be a commutative Banach algebra and let $\Delta(A)$ be the character space of A , that is, the space consisting of all non-zero homomorphisms from A into \mathbb{C} .

A bounded net $\{e_\alpha\}$ in A is called a bounded approximate identity (b.a.i) if for all $a \in A$, $\lim_\alpha ae_\alpha = a$. The notion of a bounded approximate identity first arose in Harmonic analysis; see [1, Section 2.9] for a full discussion of approximate identity and its applications.

For $a \in A$, we define $\widehat{a} : \Delta(A) \rightarrow \mathbb{C}$ by $\widehat{a}(\phi) = \phi(a)$ for all $\phi \in \Delta(A)$. Then \widehat{a} is in $C_0(\Delta(A))$ and \widehat{a} is called the Gel'fand transform of a . Note that $\Delta(A)$ is equipped with the Gel'fand topology which turns $\Delta(A)$ into a locally compact Hausdorff space; see [8, Definition 2.2.1, Theorem 2.2.3(i)]. Since for each $\phi \in \Delta(A)$, $\|\phi\| \leq 1$; see [8, Lemma 2.1.5], we have $\|\widehat{a}\|_{\Delta(A)} \leq \|a\|$, where $\|\cdot\|$ denotes the norm of A .

Suppose that $\phi \in \Delta(A)$. We denote by A_ϕ the space of all $a \in A$ such that $\text{supp } \widehat{a}$ is compact, and by J_ϕ the space of all $a \in A_\phi$ such that $\phi \notin \text{supp } \widehat{a}$. Also, let $M_\phi = \ker(\phi) = \{a \in A : \phi(a) = 0\}$.

Let X be a non-empty locally compact Hausdorff space. A subalgebra A of $C_0(X)$ is called a function algebra if A separates strongly the points of X , that is, for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$, and for each $x \in X$, there exists $f \in A$ with $f(x) \neq 0$. A function algebra A is called

a Banach function algebra if A has a norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a Banach algebra. A Banach function algebra A is called natural if $X = \Delta(A)$, that is, every character of A is an evaluation functional on some $x \in X$, or $x \rightarrow \phi_x$ is a homeomorphism. In the latter case M_{ϕ_x} is denoted by M_x . A uniform algebra on X is a Banach function algebra $(A, \|\cdot\|)$ for which $\|\cdot\|$ is equivalent to the uniform norm $\|\cdot\|_X$ of $C_0(X)$, where $\|\cdot\|_X$ defined by $\|f\|_X = \sup\{|f(x)| : x \in X\}$ for $f \in C_0(X)$.

Let S be a non-empty set. By $c_{00}(S)$, we denote the space of all functions on S of finite support. A Banach sequence algebra on S is a Banach function algebra A on S such that $c_{00}(S) \subseteq A$.

A net $\{e_\alpha\}$ in A is called a bounded weak approximate identity (b.w.a.i) if there exists a non-negative constant C such that for each α , $\|e_\alpha\| < C$ and for all $a \in A$ and $\phi \in \Delta(A)$

$$\lim_{\alpha} |\phi(ae_\alpha) - \phi(a)| = 0,$$

or equivalently, $\lim_{\alpha} \phi(e_\alpha) = 1$ for each $\phi \in \Delta(A)$. See [7] and [9] for more details.

If $\Delta(A) = \emptyset$, every bounded net in A is a b.w.a.i for A . So, to avoid trivialities we will always assume that A is a commutative Banach algebra with $\Delta(A) \neq \emptyset$.

In the case that A is a natural Banach function algebra, a bounded weak approximate identity $\{u_\alpha\}$ is called a bounded pointwise approximate identity (BPAI); see [2, Definition 2.11].

Let A be a Banach function algebra. A has bounded relative approximate units (BRAUs) of bound m if for each non-empty compact subset K of $\Delta(A)$ and $\epsilon > 0$, there exists $f \in A$ with $\|f\| < m$ and $|1 - \phi(f)| < \epsilon$ for all $\phi \in K$; see [2, Section 2.2].

In this paper, we introduce two notions of approximate identities of a Banach algebra A depending on its character space and provide some illuminating examples to show the difference of our notions from those previously known.

3 Definitions

Suppose that $\mathcal{K}(\Delta(A))$ denotes the collection of all compact subsets of $\Delta(A)$, and τ_{co} denotes the compact-open topology of $C_0(\Delta(A))$.

Definition 3.1. A *c-w approximate identity* for A (in which, by "c-w" we mean "compact-weak") is a net $\{e_\alpha\}$ in A such that for each $a \in A$ and $K \in \mathcal{K}(\Delta(A))$

$$\lim_{\alpha} \|\widehat{ae_\alpha} - \widehat{a}\|_K = 0.$$

If the net $\{e_\alpha\}$ is bounded, we say that it is a bounded c-w approximate identity (b.c-w.a.i) for Banach algebra A .

The Banach algebra A has *c-w approximate units* (c-w.a.u), if for each $a \in A$, $K \in \mathcal{K}(\Delta(A))$ and $\epsilon > 0$, there exists $e \in A$ such that $\|\widehat{ae} - \widehat{a}\|_K < \epsilon$. The approximate units have bound m , if e can be chosen with $\|e\| < m$. Clearly, if A has a b.c-w.a.i, then it has b.c-w.a.u.

Recall that $A[\tau]$ denotes the topological algebra A , where τ indicates the topology of the underlying topological vector space A ; see [3] for a general theory of topological algebras.

Definition 3.2. A net $\{a_\lambda\}$ in A is a *weakly bounded c-w approximate identity* (w.b.c-w.a.i) if the net $\{\widehat{a}_\lambda\}$ is a b.a.i for the topological algebra $\widehat{A}[\tau_{co}]$, that is, for each $a \in A$ and $K \in \mathcal{K}(\Delta(A))$, $\lim_\alpha \|\widehat{ae}_\lambda - \widehat{a}\|_K = 0$ and there exists a constant $M > 0$ such that for all λ and $K \in \mathcal{K}(\Delta(A))$,

$$(3.1) \quad P_K(\widehat{a}_\lambda) = \sup\{|\phi(a_\lambda)| : \phi \in K\} = \|\widehat{a}_\lambda\|_K < M.$$

Since, each \widehat{a}_λ is in $C_0(\Delta(A))$, Relation (3.1) is equivalent to $\|\widehat{a}_\lambda\|_{\Delta(A)} < M$.

Also, A has a *weakly bounded c-w approximate units* (w.b.c-w.a.u) of bound m , if for each $a \in A$, $K \in \mathcal{K}(\Delta(A))$ and $\epsilon > 0$, there exists $e \in A$ with $\|\widehat{e}\|_{\Delta(A)} < m$ and $\|\widehat{ae} - \widehat{a}\|_K < \epsilon$.

It is a routine calculation that each b.c-w.a.i is a w.b.c-w.a.i. We will show in the Example section that this two concepts are different.

For a natural uniform algebra A , one can see immediately that each w.b.c-w.a.i is a b.c-w.a.i.

Remark 1. We can give a more general version of Definition 3.2 as follows:

Let τ be a topology on $C_0(\Delta(A))$ generated by the saturated family (p_i) of seminorms on $\Delta(A)$; see [3, Definition 1.7] for more details of locally convex topological algebras and saturated family. A net $\{a_\alpha\}$ in A is a w.b.c-w.a.i for A if $\{\widehat{a}_\alpha\}$ is a b.a.i for the topological algebra $\widehat{A}[\tau]$, that is, for each $a \in A$, $\widehat{a}_\alpha \xrightarrow{\tau} \widehat{a}$ and there exists $M > 0$ such that for each i and α , $p_i(\widehat{a}_\alpha) < M$. But in the current paper we only focus on the case that $\tau = \tau_{co}$.

4 Examples

For a locally compact group G and $1 < p < \infty$, let $A_p(G)$ denote the Figà Talamanca-Herz algebra which is a natural Banach function algebra on G ; see

[5]. The group G is said to be amenable if there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(L_x f) = m(f)$ for each $x \in G$ and $f \in L^\infty(G)$, where $L_x f(y) = f(x^{-1}y)$; see [10, Definition 4.2]. A classical theorem due to Leptin and Herz, characterize the amenability of a group G through the existence of a bounded approximate identity for the Figà Talamanca-Herz algebra.

Now, we give the following lemma that its proof immediately follows from [2, Proposition 3.11], since each b.a.i is a b.c-w.a.i and each b.c-w.a.i is a b.w.a.i.

Lemma 4.1. *Let G be a locally compact group and $1 < p < \infty$. Then G is an amenable group if and only if $A_p(G)$ has a b.c-w.a.i.*

The following example provide for us a Banach algebra with a w.b.c-w.a.i such that has no b.c-w.a.i.

Example 1. Let $1 < p < \infty$ and G be a non-amenable locally compact group. By Lemma 4.1, $A_p(G)$ does not have any b.c-w.a.i. Now, we construct a w.b.c-w.a.i for $A_p(G)$. Put $\Lambda = \{K \subseteq G : K \text{ is compact and } |K| > 0\}$. It is obvious that Λ with inclusion is a directed set. For each $K \in \Lambda$ define u_K as follows,

$$u_K := |K|^{-1} \chi_{KK} * \check{\chi}_K.$$

Clearly, $\{u_K\}$ is a net in $A_p(G)$. For each $x \in G$ we have

$$\begin{aligned} u_K(x) &= |K|^{-1} \int_G \chi_{KK}(y) \check{\chi}_K(y^{-1}x) dy = |K|^{-1} \int_{KK} \chi_K(x^{-1}y) dy \\ &= |K|^{-1} \int_{KK} \chi_{xK}(y) dy \\ &= \frac{|KK \cap xK|}{|K|}. \end{aligned}$$

If $x \in K$, $KK \cap xK = xK$. Therefore, $u_K(x) = 1$ and otherwise since $KK \cap xK \subseteq xK$, $0 \leq u_K(x) \leq 1$. Hence, the net $\{\widehat{u_K}\}$ is bounded in $C_0(G)$.

Now, let f be an arbitrary element of $A_p(G)$ and K' be a compact subset of G . Since G is a locally compact group, for each $x \in K'$ there exists a compact neighborhood V_x of x . On the other hand, we know that $K' \subseteq \bigcup_{x \in K'} V_x$ and for each x , $|V_x| > 0$. But K' is compact, so there are points x_1, \dots, x_n in K' such that $K' \subseteq \bigcup_{i=1}^n V_{x_i}$. Therefore, by putting $K'' = \bigcup_{i=1}^n V_{x_i}$ we conclude that $K'' \in \Lambda$ and $K' \subseteq K''$. Now, it is obvious that $\lim_{K \in \Lambda} \|\widehat{u_K f} - \widehat{f}\|_{K'} = 0$ and this completes the proof.

Let G be a locally compact group, $A(G)$ be the Fourier algebra, and $L^1(G)$ be the group algebra endowed with the norm $\|\cdot\|_1$ and the convolution product. Put

$\mathfrak{L}A(G) = L^1(G) \cap A(G)$ with the norm $\|f\| = \|f\|_1 + \|f\|_{A(G)}$. We know that $\mathfrak{L}A(G)$ with the pointwise multiplication is a commutative Banach algebra called the Lebesgue-Fourier algebra of G and $\Delta(\mathfrak{L}A(G)) = G$; see [4]. It was shown that $\mathfrak{L}A(G)$ has a b.a.i if and only if G is a compact group; see [4, Proposition 2.6].

The following example provide for us a Banach algebra with a w.b.c-w.a.i whereas has no b.a.i.

Example 2. Let $G = \mathbb{R}$ be the real line additive group and $A = \mathfrak{L}A(G)$. Clearly G is not compact and hence A has no b.a.i. On the other hand, it is well-known that G is amenable, and hence $A(G)$ has a b.a.i $\{u_\alpha\}$ in $A(G) \cap C_c(G)$ by the Leptin-Herz Theorem. Also, it is well-known that for each $u \in A(G)$, $\|u\|_G \leq \|u\|_{A(G)}$. So, for each $u \in A$ and $K \in \mathcal{K}(G)$, we have

$$\sup_{x \in K} |u(x)u_\alpha(x) - u(x)| \leq \sup_{x \in G} |u(x)u_\alpha(x) - u(x)| \leq \|uu_\alpha - u\|_{A(G)} \longrightarrow 0.$$

Therefore, $\{u_\alpha\}$ is a w.b.c-w.a.i for A .

Recall that if A is a function algebra on K , then $x \in K$ is a peak point if there exists $f \in A$ such that $f(x) = 1$ and $|f(y)| < 1$ for each $y \in K \setminus \{x\}$. It is well-known that for the disc algebra $A(\overline{\mathbb{D}})$; see [1, Example 2.1.13(ii)] for more details on disc algebra, $z \in \overline{\mathbb{D}}$ is a peak point if and only if $z \in \mathbb{T}$; $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

It is worth noting that there exist Banach algebras without any w.b.c-w.a.i as the following example shows.

Example 3. Let $A = A(\overline{\mathbb{D}})$ be the disc algebra and for $z_0 \in \mathbb{D}$, let $B = M_{z_0}$. Clearly, $\overline{\mathbb{D}} \setminus \{z_0\} \subseteq \Delta(B)$. So, if B has a w.b.c-w.a.i, then B has a BPAI which is in contradiction with [2, Example 4.8(i)].

Note that if a Banach function algebra A has a b.c-w.a.i, then it has BRAUs. The following example gives a Banach function algebra with a BPAI, while it does not have any BRAUs. So, two concepts of BPAI and b.c-w.a.i are different.

Example 4. Let $\mathbb{I} = [0, 1]$ and let $A = \{f \in C(\mathbb{I}) : I(f) < \infty\}$, where

$$I(f) = \int_0^1 \frac{|f(t) - f(0)|}{t} dt.$$

For each $f \in A$, define $\|f\| = \|f\|_{\mathbb{I}} + I(f)$. By [2, Example 5.1], $(A, \|\cdot\|)$ is a natural Banach function algebra. Also, M_0 does not have BRAUs, but it has a BPAI. Therefore, M_0 does not have any b.c-w.a.i.

The following example shows the difference between b.a.i and b.c-w.a.i.

Example 5. Let $\alpha = (\alpha_k) \in \mathbb{C}^{\mathbb{N}}$ and for each $n \in \mathbb{N}$, set

$$p_n(\alpha) = \frac{1}{n} \sum_{k=1}^n k |\alpha_{k+1} - \alpha_k|, \quad p(\alpha) = \sup_{n \in \mathbb{N}} p_n(\alpha).$$

Put $A = \{\alpha \in c_0 : p(\alpha) < \infty\}$. By [1, Example 4.1.46], A is a natural Banach sequence algebra on \mathbb{N} for the norm given by $\|\alpha\| = \|\alpha\|_{\mathbb{N}} + p(\alpha)$. For each $K \in \mathcal{K}(\mathbb{N})$ there exists $\alpha_K \in A$ such that $\alpha_K(k) = 1$ for all $k \in K$ and $\|\alpha_K\| \leq 4$. So, $\{\alpha_K : K \in \mathcal{K}(\mathbb{N})\}$ is a b.c-w.a.i for A . But by parts (iii) and (v) of [1, Example 4.1.46], A^2 has infinite codimension in A where $A^2 = \text{linear span}\{ab : a, b \in A\}$. Therefore, A has no b.a.i.

Remark 2. By Cohen's factorization Theorem, we know that if a Banach algebra A has a b.a.i, then A factors, that is, for each $a \in A$ there exists $b, c \in A$ such that $a = bc$. One may ask this question: Whether the statement of the Cohen Theorem is valid if we replace b.a.i by b.c-w.a.i (w.b.c-w.a.i)? Example 5 gives a negative answer to this question. Indeed, if A factors, then $A^2 = A$ which is a contradiction.

Let G be a locally compact abelian group and let $A = S(G)$ be a Segal algebra in $L^1(G)$; see [1, Definition 4.5.26] or [11]. By using the Cohen factorization Theorem, if A has a b.a.i, then $A = L^1(G)$. But with using the Šilov idempotent Theorem we show in the sequel that there exist a Segal algebra $S(G)$ with a w.b.c-w.a.i such that does not satisfy $S(G) = L^1(G)$.

First we recall the Šilov idempotent Theorem as follows; see [8, Theorem 3.5.1] or [1, Theorem 2.4.33].

Theorem 4.2. *Let A be a commutative Banach algebra and C be a compact and open subset of $\Delta(A)$. Then there exists an idempotent $a \in A$ such that \widehat{a} is equal to the characteristic function of C .*

Proposition 4.3. *Every commutative Banach algebra A with discrete character space has a w.b.c-w.a.i. But the converse is not valid in general.*

Proof. Let $\mathcal{F}(\Delta(A))$ be the collection of all finite subsets of $\Delta(A)$ and let K be an element of $\mathcal{F}(\Delta(A))$. So, by using Šilov's idempotent Theorem, we can take an element e_K in A such that $\widehat{e_K} = \chi_K$. So, for each $K \in \mathcal{F}(\Delta(A))$, $\|\widehat{e_K}\|_{\Delta(A)} = 1$.

Now, it is clear that $\{e_K : K \in \mathcal{F}(\Delta(A))\}$ is a w.b.c-w.a.i for A , where $\{K : K \in \mathcal{F}(\Delta(A))\}$ is ordered with inclusion. Because for each $a \in A$ and $K' \in \mathcal{F}(\Delta(A))$, $\lim_K \|\widehat{e_K a} - \widehat{a}\|_{K'} = 0$.

To see that the converse is not valid consider $A_p(G)$, where $G = SL(2, \mathbb{R})$ is the multiplicative group of all 2×2 real matrices with determinant 1. We know that G is non-amenable; see [12, Exercise 1.2.6 (viii)], and by [13, Proposition 1.4, pp. 207], G is a connected group and hence it is not discrete. By Example 1, $A_p(G)$ has a w.b.c-w.a.i but $\Delta(A_p(G)) = G$ is not discrete. \square

Example 6. Let G be a locally compact abelian group with dual group \widehat{G} . For each $1 < p \leq \infty$, we put $S_p(G) = L^1(G) \cap L^p(G)$ and define a norm as

$$\|f\|_{S_p(G)} = \max\{\|f\|_1, \|f\|_p\} \quad (f \in S_p(G)).$$

Then $S_p(G)$ is a Segal algebra. If G is a compact and infinite group, then by [6, Remark 2], $S_p(G)$ has no b.w.a.i. Let $A = S_p(G)$. We know that $\Delta(A)$ is homeomorphic to \widehat{G} ; see [11]. Since G is a compact group, it is well-known that \widehat{G} is discrete. So, as an application of Proposition 4.3, we see that A has a w.b.c-w.a.i.

5 Hereditary Properties

In this section we will show that for some certain closed ideals of a Banach algebra A with some conditions, I has a b.c-w.a.i (w.b.c-w.a.i) if and only if A has a b.c-w.a.i (w.b.c-w.a.i). First, we give the following result which shows the relation between b.c-w.a.u (w.b.c-w.a.u) and b.c-w.a.i (w.b.c-w.a.i), and it is a key tool in the sequel.

Proposition 5.1. *Let A be a commutative Banach algebra. Then A has a b.c-w.a.u (w.b.c-w.a.u) if and only if A has a b.c-w.a.i (w.b.c-w.a.i).*

Proof. Let A has a b.c-w.a.u of bound $M > 0$. Suppose that \mathcal{F} is a finite subset of A , $K \in \mathcal{K}(\Delta(A))$ and $\epsilon > 0$. Then like the proof of [8, Proposition 1.1.11], there exists $e_{(\mathcal{F}, K, \epsilon)} \in A$ such that $\|e_{(\mathcal{F}, K, \epsilon)}\| < M$ and

$$\|a\widehat{e_{(\mathcal{F}, K, \epsilon)}} - \widehat{a}\|_K < \epsilon \quad (a \in \mathcal{F}).$$

Now, consider the following net,

$$\mathfrak{U} = \{e_{(\mathcal{F}, K, 1/n)} : K \in \mathcal{K}(\Delta(A)), \mathcal{F} \subseteq A \text{ is finite and } n \in \mathbb{N}\}.$$

It is a routine calculation that \mathfrak{U} is a b.c-w.a.i for A . \square

Let I be a closed ideal of A . Suppose that I and A/I ; the quotient Banach algebra, respectively have b.a.i of bound m and n . Then A has a b.a.i of bound $m + n + mn$; see [8, Lemma 1.4.8 (ii)]. In the setting of b.c-w.a.i we have the following version of the mentioned assertion.

Lemma 5.2. *Suppose that I has a w.b.c-w.a.i (b.c-w.a.i) of bound m and A/I has a b.a.i of bound m . Then A has a w.b.c-w.a.i (b.c-w.a.i) of bound $m+n+mn$.*

Proof. With a slight modification in the proof of [8, Lemma 1.4.8 (ii)], we can see the proof. Therefore, we omit the details. \square

There exists a Banach function algebra A such that A has a b.c-w.a.i, but one of its closed ideals has no b.c-w.a.i; see [9, Example 5.6]. Indeed, $A = C^1[0, 1]$; the algebra of all functions with continuous derivation, has a b.c-w.a.i, but for each $t_0 \in [0, 1]$, M_{t_0} has no BPAI and hence no b.c-w.a.i. So, the converse of Lemma 5.2 is not valid in general. Although we have the following result which its proof is a mimic of [2, Proposition 2.12 (i)].

Theorem 5.3. *Let A be a Banach algebra, $\phi_0 \in \Delta(A)$ and $\overline{J_{\phi_0}} = M_{\phi_0}$. Suppose that there exists $n > 0$ such that for each neighborhood U of ϕ_0 , there exists $a \in A$ with $\phi_0(a) = 1$, $\|a\| \leq n$ and $\text{supp } \widehat{a} \subseteq U$. Then M_{ϕ_0} has a w.b.c-w.a.i (b.c-w.a.i) if and only if A has a w.b.c-w.a.i (b.c-w.a.i).*

Proof. We only give the proof for the case "w.b.c-w.a.i", since the proof for "b.c-w.a.i" is similar.

Suppose that M_{ϕ_0} has a w.b.cw-a.i. Clearly, M_{ϕ_0} has codimension 1, that is, A/M_{ϕ_0} is generated by one vector. Therefore, A/M_{ϕ_0} has a b.a.i and hence by Lemma 5.2, A has a w.b.c-w.a.i.

Conversely, suppose that A has a w.b.c-w.a.i. Let $a \in M_{\phi_0}$ and $\varepsilon > 0$. Since $\overline{J_{\phi_0}} = M_{\phi_0}$, there exists $a_1 \in J_{\phi_0}$ such that

$$\|a - a_1\| < \varepsilon.$$

Therefore, for each $W \in \mathcal{K}(\Delta(A))$, there exists $m > 0$ and $b \in A$ such that

$$\|\widehat{a_1} - \widehat{a_1 b}\|_W < \varepsilon, \quad \|\widehat{b}\|_{\Delta(A)} \leq m.$$

There exists a neighborhood U of ϕ_0 in $\Delta(A)$ such that $U \cap \text{supp } \widehat{a_1} = \emptyset$, because $a_1 \in J_{\phi_0}$ and $\Delta(A)$ is Hausdorff. Now, by the hypothesis, there exists $c \in A$ such that $\|c\| \leq n$, $\phi_0(c) = 1$ and $\text{supp } \widehat{c} \subseteq U$. Therefore, $\widehat{a_1 c} = 0$, $b - bc$ is in M_{ϕ_0} and $\|\widehat{b - bc}\|_{\Delta(A)} \leq m(n+1)$. So, for $W \in \mathcal{K}(\Delta(A))$ we have

$$\begin{aligned} \|\widehat{a} - \widehat{a(b - bc)}\|_W &\leq \|\widehat{a} - \widehat{a_1}\|_W + \|\widehat{a_1} - \widehat{a_1 b}\|_W + \|\widehat{a_1 b} - \widehat{a(b - bc)}\|_W \\ &\leq \|a - a_1\| + \varepsilon + \|\widehat{a_1 b} - \widehat{a(b - bc)} + \widehat{b a_1 c}\|_W \\ &\leq 2\varepsilon + \varepsilon m(1 + n). \end{aligned}$$

Hence, by Proposition 5.1, M_{ϕ_0} has a w.b.c-w.a.i, which completes the proof. \square

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